2. (14%) Find the general solution of $y''+2y'+y=e^{-t}\ln t$

1. (21%) Solve completely the following second-order linear equations:'
   a. $2y''-7y'+3y=0$.
   b. $y''+8y'+25y=0;\ y(0)=1;\ y'(0)=0$.
   c. $y''-9y=6e^{3t}$

(You may use the back of the page to complete this problem)

3. (10%) Give only the <u>general form</u> of the general solution to:

$$y'' - y' - 12y = \cos(2t) + 3e^{4t}$$

4. Given that $y_1(t) = t$ is a solution for $(t^2 - 1)y'' - 2ty' + 2y = 0; 0 \le t < 1$,

   a. (10%) Find its general solution

b. (2%) Show that this solution is actually valid for all t.

5. (13%) If the Wronskian of two functions $f(t)$ and $g(t)$ is $t^2 e^t$ and if $f(t) = t$, find $g(t)$. You may need to recall the form of the solution to a first-order linear differential equation: $y' + p(t)y = q(t)$. It takes the form:

$$y(t) = \frac{1}{\mu(t)} \int \mu(t) q(t) dt, \text{ where } \mu(t) = \int e^{p(t)dt}$$

6. (10%) Determine the value or values of α, if any, for which all solutions to the differential equation $y''-(2\alpha-1)y'+\alpha(\alpha-1)y=0$ tend to zero as $t\to\infty$.

7. The following is a <u>third-order</u> Euler homogeneous differential equation:

$$x^3y'''+6x^2y''+7xy'+y=0; x>0$$

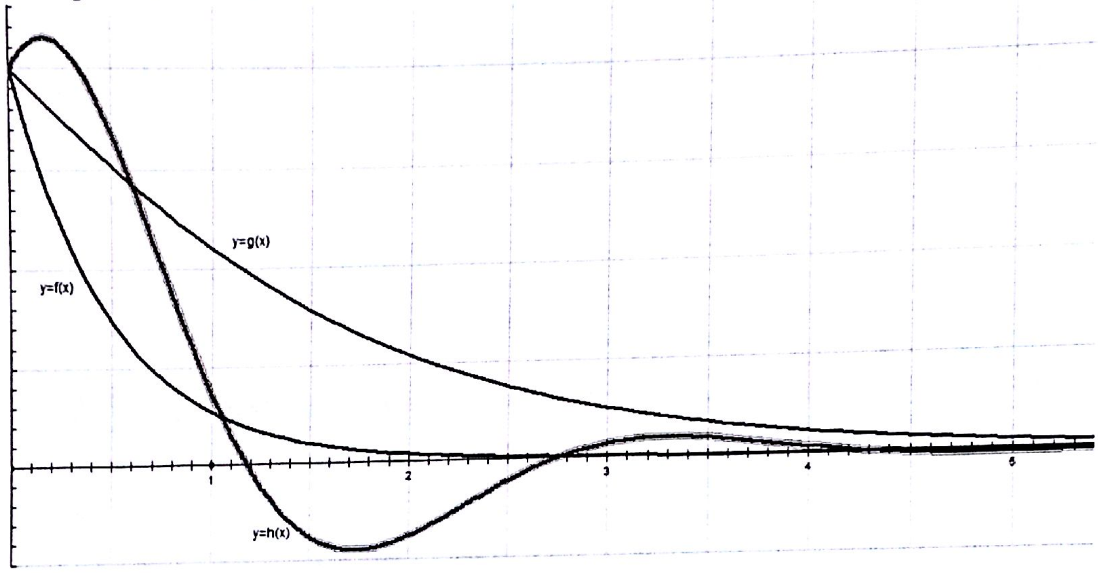
Let $x=e^t$.

a. (5%) Show: $\begin{cases} \dfrac{dy}{dx}=\dfrac{1}{x}\dfrac{dy}{dt} \\ \dfrac{d^2y}{dx^2}=-\dfrac{1}{x^2}\dfrac{dy}{dt}+\dfrac{1}{x^2}\dfrac{d^2y}{dt^2} \\ \dfrac{d^3y}{dt^3}=\dfrac{2}{x^3}\dfrac{dy}{dt}-\dfrac{3}{x^3}\dfrac{d^2y}{dt^2}+\dfrac{1}{x^3}\dfrac{d^3y}{dt^3} \end{cases}$

(Use next page to solve this problem)

8. (8%) Writing Assignment: In class we discussed few times doors and how they would close. The modeling of such a problem is through a second order differential equation with constant coefficients that are positive. Some solutions take one of the following forms: $f(t) = Ae^{-at}; g(t) = Ae^{-at} + Bte^{-at}; a, b, A, B > 0$, while others take the form $h(t) = e^{-at}\left(k_1 \cos(bt) + k_2 \sin(bt)\right)$

Below are sample graphs of such functions; discuss the differences in which these doors actually come to rest. Also discuss what happens in case $h(t) = \cos(bt)$ i.e. $a = k_2 = 0$. Is there a best model for this problem?



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$(2-t)y''' + (2t-3)y'' - ty' + y = 0; \; t<2$

Given that $y_1(t) = e^t$ is one solution, find $y_2(t)$.

Solution: $y_2(t) = v(t)y_1(t) \Rightarrow y_2' = v'y_1 + vy_1'$; $y_2'' = v''y_1 + v'y_1' + v'y_1' + vy_1''$

ie $y_2'' = v''y_1 + 2v'y_1' + vy_1''$.

$y_2''' = v'''y_1 + v''y_1' + 2v''y_1' + 2v'y_1'' + v'y_1'' + vy_1'''$

ie $y_2''' = v'''y_1 + 3v''y_1' + 3v'y_1'' + vy_1'''$.

We now substitute in the ODE:

$(2-t)[v'''y_1 + 3v''y_1' + 3v'y_1'' + vy_1'''] + (2t-3)[v''y_1 + 2v'y_1' + vy_1''] - t[v'y_1 + vy_1'] + vy_1 = 0$

(= 0 since y_1 is a solution)

$\Rightarrow (2-t)vy_1''' + (2t-3)vy_1'' - tvy_1' + vy_1 + (2-t)$
$+ (2-t)[v'''y_1 + 3v''y_1' + 3v'y_1''] + (2t-3)[v''y_1 + 2v'y_1'] - tv'y_1 = 0$

$\therefore (2-t)[v'''y_1 + 3v''y_1' + 3v'y_1''] + (2t-3)[v''y_1 + 2v'y_1'] - tv'y_1 = 0.$

Since $y_1 = e^t \Rightarrow y_1' = e^t$ and $y_1'' = e^t$ $\therefore$ e^t is a common factor to all terms; we can remove it since $e^t \neq 0$

$\therefore (2-t)[v''' + 3v'' + 3v'] + (2t-3)(v'' + 2v') - tv' = 0$

$\Rightarrow (2-t)v''' + (3-t)v'' = 0$.

Let $w = v'' \Rightarrow w' = v''' \Rightarrow (2-t)w' + (3-t)w = 0$

This is a 1st order separable ODE: $\frac{w'}{w} = -\frac{3-t}{2-t} = -\left[1 + \frac{2}{2-t}\right]$

$\therefore \int \frac{w'}{w} dt = \int \left(-1 + \frac{2}{t-2}\right) dt = -t + 2\ln|t-2|$

$\therefore \ln|w| = -t + 2\ln|t-2| \Rightarrow w = Ce^{-t+2\ln|t-2|} = Ce^{-t} \cdot e^{2\ln|t-2|}$

$\Rightarrow w = C(t-2)^2 e^{-t}$ (choose $C = 1$)

$\Rightarrow v'' = (t-2)^2 e^{-t} \Rightarrow v' = \int (t-2)^2 e^{-t} dt \leftarrow$ to be done by parts

--- etc ...